Derivation of Invariants by Tensor Methods

Tomáš Suk

Motivation

Invariants to geometric transformations of 2D and 3D images

Tensor Calculus

William Rowan Hamilton, On some extensions of Quaternions, Philosophical Magazine (4th series):
vol. vii (1854), pp. 492-499,
vol. viii (1854), pp. 125-137, 261-9,
vol. ix (1855), pp. 46-51, 280-290.

Gregorio Ricci, Tullio Levi-Civita, Méthodes de calcul différentiel absolu et leurs applications, Mathematische Annalen (Springer) 54 (1–2): pp. 125–201, March 1900.

Tensor Calculus

Grigorii Borisovich Gurevich, Osnovy teorii algebraicheskikh invariantov, (OGIZ, Moskva), 1937.
Foundations of the Theory of Algebraic Invariants, (Nordhoff, Groningen), 1964.

David Cyganski, John A. Orr, Object Recognition and Orientation Determination by Tensor Methods, in *Advances in Computer Vision and Image Processing*, JAI Press, pp. 101—144, 1988.

Tensor Calculus

Tensor = multidimensional array

+ rules of multiplication

2 rules:

- Enumeration
- Sum of products (dot product)

Example: product of 2 vectors

Sum of products: $$\begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = c$$

Tensor Calculus

Example: product of 2 vectors $a_i b_k = c_{ik}$

Enumeration:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix} =$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & & & \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Tensor Calculus

Example: product of 2 matrices $a_{ij}b_{kl}$

I enumeration, j=k sum of products, l enumeration

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n_1} \\ \vdots & & & \\ \mathbf{a_{i1}} & \mathbf{a_{i2}} & \cdots & \mathbf{a_{in_1}} \\ \vdots & & & \\ a_{n_2 1} & a_{n_2 2} & \cdots & a_{n_2 n_1} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & \mathbf{b_{1\ell}} & \cdots & b_{1n_3} \\ b_{21} & \cdots & \mathbf{b_{2\ell}} & \cdots & b_{2n_3} \\ \vdots & & & & \\ b_{n_1 1} & \cdots & \mathbf{b_{n_1\ell}} & \cdots & b_{n_1 n_3} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & \cdots & c_{1\ell} & \cdots & c_{1n_1} \\ \vdots & & & & \\ c_{i1} & \cdots & \mathbf{c_{i\ell}} & \cdots & c_{in_1} \\ \vdots & & & & \\ c_{n_2 1} & \cdots & c_{n_2\ell} & \cdots & c_{n_2 n_3} \end{pmatrix}$$

Einstein Notation

Instead of

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad \forall i, k = 1, \ldots, n$$

we write

$$c_{ik} = a_{ij} b_{jk}$$

other options

vectors: $a_i b_i, \; a_i b_j$

matrices:

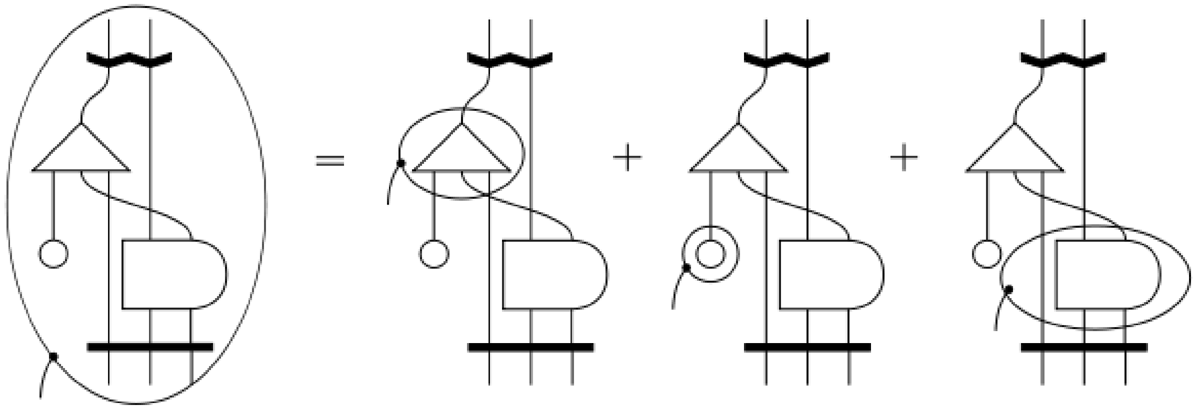
$$a_{ij} b_{ij}, \; a_{ij} b_{ji}, \; a_{ij} b_{kj}, \; a_{ij} b_{ik}, \; a_{ij} b_{ki}, \; a_{ij} b_{k\ell}$$

Other Notations

Other symbols

$$\otimes \quad \bullet \quad \times_{(n)} \quad *$$

Penrose graphical notation



Tensors in Affine Space

Contravariant vector

$$x^i = p^i_\alpha \hat{x}^\alpha \qquad \hat{x}^\alpha = q^\alpha_i x^i$$

Covariant vector

$$\hat{u}_\alpha = p^i_\alpha u_i$$

Affine transform in 2D

$$\mathbf{A} = \begin{pmatrix} p^1_1 & p^1_2 \\ p^2_1 & p^2_2 \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} q^1_1 & q^1_2 \\ q^2_1 & q^2_2 \end{pmatrix}$$

Tensors in Affine Space

Contravariant tensor

$$a^{i_1,i_2,\ldots,i_r} = p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_r}_{\alpha_r} \hat{a}^{\alpha_1,\alpha_2,\ldots,\alpha_r}$$

Covariant tensor

$$\hat{a}_{\alpha_1,\alpha_2,\ldots,\alpha_r} = p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_r}_{\alpha_r} a_{i_1,i_2,\ldots,i_r}$$

Mixed tensor

$$\hat{a}^{\beta_1,\beta_2,\ldots,\beta_{r_2}}_{\alpha_1,\alpha_2,\ldots,\alpha_{r_1}} = q^{\beta_1}_{j_1} q^{\beta_2}_{j_2} \cdots q^{\beta_{r_2}}_{j_{r_2}} p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_{r_1}}_{\alpha_{r_1}} a^{j_1,j_2,\ldots,j_{r_2}}_{i_1,i_2,\ldots,i_{r_1}}$$

Tensors in Affine Space

Multiplication

$$a^i_j b^j_k = c^i_k$$

Adition

$$a^i_j + b^i_j = c^i_j$$

Geometric Meaning

Point - contravariant vector $$x^i = p^i_\alpha \hat{x}^\alpha$$

Angle of straight line – covariant vector

$$u_i x^i = 0 \qquad \hat{u}_i = p^i_\alpha u^\alpha$$

Conic section – covariant tensor

$$a_{ij} x^i x^j + 2 b_i x^i + h = 0 \qquad \hat{a}_{ij} = p^i_\alpha p^j_\beta a^{\alpha\beta}$$

Properties

Symmetric tensor

-To two indices $a_{ijk} = a_{kji}$

-To several indices

-To all indices

Antisymmetric tensor (skew-symmetric)

-To two indices $a_{ijk} = -a_{kji}$

-To several indices

-To all indices

Other Tensor Operations

Symmetrization

$$a_{(ijk)} = \frac{1}{6}(a_{ijk} + a_{ikj} + a_{jki} + a_{jik} + a_{kij} + a_{kji})$$
$$= \frac{1}{6}\sum_{p\in P} a_{\{ijk\}_p}$$

Alternation (antisymmetrization)

$$a_{[ijk]} = \frac{1}{6}(a_{ijk} - a_{ikj} + a_{jki} - a_{jik} + a_{kij} - a_{kji})$$

$$h_{[i|j|k]} = \frac{1}{2}(h_{ijk} - h_{kji})$$

Other Tensor Operations

Contraction

$$k^{i}_{\ell m} = h^{ij}_{j\ell m} \qquad\qquad h^{ij}_{ijm},\ h^{ij}_{jim},\ h^{ij}_{kij},$$

Total contraction

$$a^{ij}_{k\ell} \longrightarrow a^{ij}_{ij},\ a^{ij}_{ji}$$

$\rightarrow$ Affine invariant

$$\hat{a}^{\beta_1,\beta_2,\ldots,\beta_{r_2}}_{\alpha_1,\alpha_2,\ldots,\alpha_{r_1}} = q^{\beta_1}_{j_1} q^{\beta_2}_{j_2} \cdots q^{\beta_{r_2}}_{j_{r_2}} p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_{r_1}}_{\alpha_{r_1}} a^{j_1,j_2,\ldots,j_{r_2}}_{i_1,i_2,\ldots,i_{r_1}}$$

Unit polyvector

Also Levi-Civita symbol

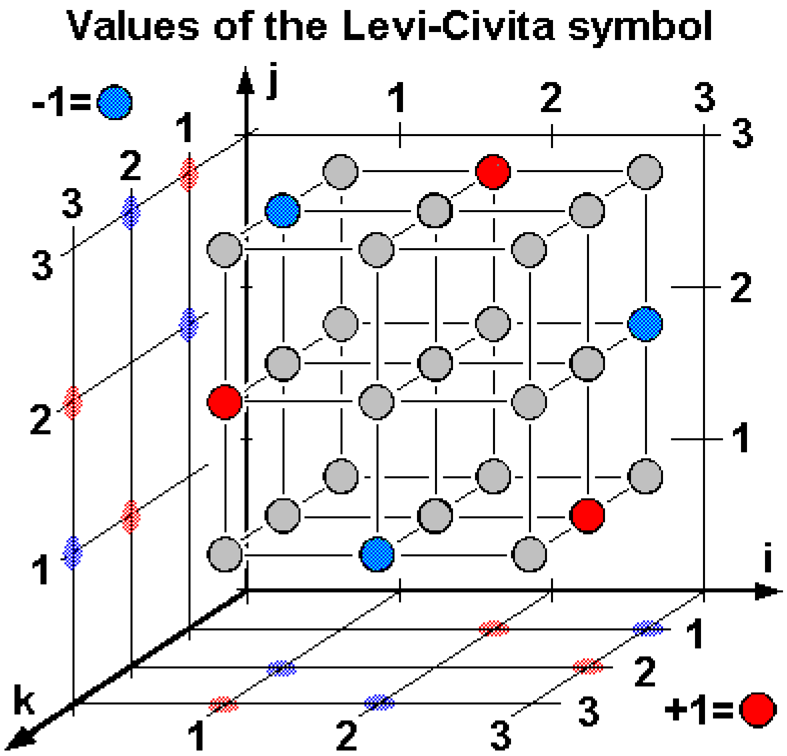
Completely antisymmetric tensor

- covariant $\epsilon_{12\ldots n} = 1$
- contravariant $\epsilon^{12\ldots n} = 1$

n=2: $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Unit polyvector

n=3:



n^n components

n! non-zero

note: vector cross product $\overrightarrow{a} \times \overrightarrow{b} = \epsilon_{ijk} a^j b^k$

Affine Invariants

- Products with total contraction

$$a^{ijk}b_{ijk}$$

- Using unit polyvectors

$$a^{ijk}b^{\ell mn}\epsilon_{ij}\epsilon_{k\ell}\epsilon_{mn}$$

$$a_{ijk}b_{\ell mn}\epsilon^{i\ell}\epsilon^{jm}\epsilon^{kn}$$

Example - moments

2D Moment tensor

$$M^{i_1 i_2 \cdots i_r} = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x^{i_1} x^{i_2} \cdots x^{i_r} f(x^1, x^2) \, \mathrm{d}x^1 \, \mathrm{d}x^2$$

$M^{i_1 i_2 \cdots i_r} = m_{pq}$ if p indices equals 1 and q indices equals 2

Geometric moment

$$m_{pq} = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x^p y^q f(x, y) \mathrm{d}x \mathrm{d}y$$

Example - moments

Relative oriented contravariant tensor with weight -1

$$\hat{M}^{i_1 i_2 \cdots i_r} = |J|^{-1} q^{i_1}_{\alpha_1} q^{i_2}_{\alpha_2} \cdots q^{i_r}_{\alpha_r} M^{\alpha_1 \alpha_2 \cdots \alpha_r}$$

Example - moments

$$M^{ij}M^{klm}M^{nop}\epsilon_{ik}\epsilon_{jn}\epsilon_{lo}\epsilon_{mp} =$$

$$= 2(m_{20}(m_{21}m_{03} - m_{12}^2) -$$
$$-m_{11}(m_{30}m_{03} - m_{21}m_{12}) +$$
$$+m_{02}(m_{30}m_{12} - m_{21}^2))$$

$$\rightarrow \quad (2\mu_{20}\mu_{21}\mu_{03} - 2\mu_{20}\mu_{12}^2 -$$
$$-\mu_{11}\mu_{30}\mu_{03} + \mu_{11}\mu_{21}\mu_{12} +$$
$$+\mu_{02}\mu_{30}\mu_{12} - \mu_{02}\mu_{21}^2)/\mu_{00}^7$$

Example – 3D moments

$$M^{i_1 i_2 \cdots i_k} =$$

$$= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x^{i_1} x^{i_2} \cdots x^{i_k} f(x^1, x^2, x^3)\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3$$

$$M^{ij} M^{kl} M^{mn} \epsilon_{ikm} \epsilon_{jln} =$$

$$= 6(m_{200} m_{020} m_{002} + 2 m_{110} m_{101} m_{011} -$$

$$- m_{200} m_{011}^2 - m_{020} m_{101}^2 - m_{002} m_{110}^2)$$

Invariants to other transformations

- Non-linear transforms

Jacobi matrix

$$\mathbf{A} = \begin{pmatrix} \frac{\partial g^1(x^1,x^2)}{\partial x^1} & \frac{\partial g^1(x^1,x^2)}{\partial x^2} \\ \frac{\partial g^2(x^1,x^2)}{\partial x^1} & \frac{\partial g^2(x^1,x^2)}{\partial x^2} \end{pmatrix}$$

Invariants to other transformations

- Projective transform

$$x = \frac{p_1^1 \hat{x} + p_2^1 \hat{y} + p_3^1}{p_1^3 \hat{x} + p_2^3 \hat{y} + p_3^3}$$

$$y = \frac{p_1^2 \hat{x} + p_2^2 \hat{y} + p_3^2}{p_1^3 \hat{x} + p_2^3 \hat{y} + p_3^3}$$

Homogeneous coordinates

$$x = \frac{x^1}{x^3} \qquad y = \frac{x^2}{x^3}$$

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} p_1^1 & p_2^1 & p_3^1 \\ p_1^2 & p_2^2 & p_3^2 \\ p_1^3 & p_2^3 & p_3^3 \end{pmatrix} \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \end{pmatrix}$$

Rotation Invariants

- Cartesian tensors

Affine tensor

$$\hat{a}_{\alpha_1,\alpha_2,\ldots,\alpha_{r_1}}^{\beta_1,\beta_2,\ldots,\beta_{r_2}} = q_{j_1}^{\beta_1} q_{j_2}^{\beta_2} \cdots q_{j_{r_2}}^{\beta_{r_2}} p_{\alpha_1}^{i_1} p_{\alpha_2}^{i_2} \cdots p_{\alpha_{r_1}}^{i_{r_1}} a_{i_1,i_2,\ldots,i_{r_1}}^{j_1,j_2,\ldots,j_{r_2}}$$

Cartesian tensor

$$\hat{a}_{\alpha_1,\alpha_2,\ldots,\alpha_r} = p_{\alpha_1 i_1} p_{\alpha_2 i_2} \cdots p_{\alpha_r i_r} a_{i_1,i_2,\ldots,i_r}$$

Orthonormal matrix $p_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Cartesian Tensors

- Rotation invariants by total contraction

$$a_{ijk}b_{ijk}$$

Rotation Invariants

$$M_{ii} \;\longrightarrow$$

2D: $$M_{11} + M_{22} \;\longrightarrow\; (\mu_{20} + \mu_{02})/\mu_{00}^{2}$$

3D: $$M_{11} + M_{22} + M_{33} \;\longrightarrow$$
$$(\mu_{200} + \mu_{020} + \mu_{002})/\mu_{000}^{5/3}$$

Rotation Invariants

$$M_{ij}M_{ij} \longrightarrow$$

2D: $$M_{11}M_{11} + M_{12}M_{12} + M_{21}M_{21} + M_{22}M_{22}$$

$$\longrightarrow (\mu_{20}^2 + \mu_{02}^2 + 2\mu_{11}^2)/\mu_{00}^4$$

3D: $$M_{11}M_{11} + M_{12}M_{12} + M_{13}M_{13} +$$

$$+ M_{21}M_{21} + M_{22}M_{22} + M_{23}M_{23} +$$

$$+ M_{31}M_{31} + M_{32}M_{32} + M_{33}M_{33}$$

$$\longrightarrow (\mu_{200}^2 + \mu_{020}^2 + \mu_{002}^2 + 2\mu_{110}^2 +$$

$$+ 2\mu_{101}^2 + 2\mu_{011}^2)/\mu_{000}^{10/3}$$

Rotation Invariants

$$M_{ij}M_{jk}M_{ki} \longrightarrow$$

2D: $$(\mu_{20}^3 + 3\mu_{20}\mu_{11}^2 + 3\mu_{11}^2\mu_{02} + \mu_{02}^3)/\mu_{00}^6$$

3D: $$\begin{aligned}&(\mu_{200}^3 + 3\mu_{200}\mu_{110}^2 + 3\mu_{200}\mu_{101}^2 + \\ &+3\mu_{110}^2\mu_{020} + 3\mu_{101}^2\mu_{002} + \mu_{020}^3 + \\ &+3\mu_{020}\mu_{011}^2 + 3\mu_{011}^2\mu_{002} + \mu_{002}^3 + \\ &+6\mu_{110}\mu_{101}\mu_{011})/\mu_{000}^5\end{aligned}$$

Alternative Methods

- Method of geometric primitives

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_1 y_2 - x_2 y_1)^2 f(x_1, y_1) f(x_2, y_2) \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2$$

- Solution of Equations

  Transformation decomposition
  System of linear equations for unknown coefficients

Alternative Methods

- Complex moments

2D: $$c_{pq} = \int\limits_0^\infty \int\limits_0^{2\pi} r^{p+q+1} e^{-i(p-q)\alpha} f(r,\theta)\mathrm{d}r\mathrm{d}\theta$$

3D: $$c_{s\ell}^{m} = \int\limits_0^\infty \int\limits_0^{\pi} \int\limits_0^{2\pi} r^{s+1} Y_\ell^m(\vartheta,\varphi) f(r,\vartheta,\varphi)\mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi$$

- Normalization

Transformation decomposition

Thank you for your attention